Homework 3: Linear Algebra Basics

Due: 30-Jan-2017, beginning of class

Linear vs. non-Linear terms

Let the variables x_i (where i is an integer) denote independent variables. All other variables are constants. Next to each term write down whether it is linear or non-linear:

1. $c^2x_1 - x_2$
2. $\sin(x_1) + 5x_1$
3. x_1/x_2
4. $ax_1 - bx_2$
5. $\tan(ax_1 - bx_2)$
6. $(x_1 - 1)(x_2 - 2)$
7. $x_1 + x_2 + x_3$
8. $x_3 - x_1 + x_2x_1$

Writing a system of linear equations in matrix form

The system of linear equations

$$\begin{aligned} 3x_2 + &= -1 \\ x_1 + 6x_2 &= 2 \end{aligned}$$

is written in matrix form $\mathbf{Ax} = \mathbf{b}$ as

$$\underbrace{\begin{pmatrix} 0 & 3 \\ 1 & 6 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{\mathbf{b}}$$

where the $\mathbf{A}$ matrix has $n = 2$ rows and $m = 2$ columns, thus: $\mathbf{A}$ is a (2×2) matrix.

Write each of the following system of equations in the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a matrix, and $\mathbf{x}$ is a column vector of the x_i independent variables, and $\mathbf{b}$ is a column vector of constants. State the dimensions of each $\mathbf{A}$ matrix (i.e., if it has m rows and n columns then write "$\mathbf{A}$ is a $(m \times n)$ matrix."

9.

$$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

10.

$$\begin{aligned} 3x_2 - x_3 &= 4 \\ x_1 + 2x_2 + x_3 &= 3 \\ x_1 + 4x_3 &= 0 \end{aligned}$$

11.

$$\begin{aligned} 3x_3 - x_2 &= 4 \\ x_3 + 2x_1 + x_2 &= 3 \\ x_3 + 4x_1 &= 0 \end{aligned}$$

12.

$$\begin{aligned} 2x_1 - x_2 - x_4 &= 7 \\ -x_1 + 2x_2 - x_3 &= 4 \\ -x_2 + 2x_3 - x_4 &= 10 \\ -x_1 - x_3 + 2x_4 &= 0.5 \end{aligned}$$

Writing a matrix equation in standard form

The matrix equation

$$\begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is written in standard form as

$$\begin{aligned} x_1 &= 3 \\ 4x_1 - x_2 &= 2 \end{aligned}$$

Write each of the following matrix equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ in standard form:

13.

$$\begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

14.

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

15.

$$\begin{pmatrix} 5 & 3 & 0 \\ 5 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

16.

$$\begin{pmatrix} 3 & 0 & 0.1 & 0 \\ -5 & -3 & 1 & -5.5 \\ 0 & 6.5 & 0 & 1 \\ 1 & -8 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 9 \\ 100 \end{pmatrix}$$

Matrix addition, subtraction, scalar multiplication, and matrix transpose

The matrices

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 6 & 3 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{E} = \begin{pmatrix} 3 & 1 & 2 \\ -4 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

are entered into MATLAB as

```
D = [1 0 0; -2 6 3; 0 1 0];
E = [3 1 2; -4 2 3; 1 0 1];
```

Matrix multiplication is implemented in MATALB using the `*` symbol. The matrix expression $\mathbf{2DE} + (\mathbf{D} - \mathbf{E})$ is written in MATLAB as `2*D*E+(D-E)`. The matrix transpose $\mathbf{D}^{\mathrm{T}}$ exchanges the rows and columns of a matrix, so that:

$$\mathbf{D}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 6 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

Matrix transposition is implemented in MATALB using the `'` symbol adjacent to a matrix (i.e., $\mathbf{D}^{\mathrm{T}}$ is written `D'`). Use MATLAB to evaluate the following matrix expressions, where:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 6 & 1 & -1 \\ 0 & 1 & 0 & 3 \\ 4 & 6 & 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 4 & 1 & 7 \\ 2 & -1 & 0 & 0 \\ 10 & 0 & 0 & 1 \end{pmatrix}$$

13. $\mathbf{A} - \mathbf{C}$
14. $\mathbf{A} + \mathbf{3C}$
15. $\mathbf{2}(\mathbf{2A} - \mathbf{C})$
16. $\mathbf{A} - \mathbf{AC}^{\mathrm{T}}\mathbf{A}$
17. Does the order of multiplication matter? (i.e., is $\mathbf{AC}$ the same as $\mathbf{CA}$?)
18. What is the result of $\mathbf{A} - \mathbf{A}$?
19. What do you notice about the result when a matrix is multiplied by its transpose (e.g., $\mathbf{AA}^{\mathrm{T}}$, $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ or $\mathbf{CC}^{\mathrm{T}}$)?

Matrix-by-vector multiplication

A common matrix operation we will use in this course is to pre-multiply a vector by a matrix to produce another vector. This is an equation of the form $\mathbf{y} = \mathbf{A}\mathbf{x}$ where both $\mathbf{y}$ and $\mathbf{x}$ are column vectors, and $\mathbf{A}$ is a matrix.

Let

$$\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 6 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then in MATLAB syntax,

```
A = [1 0 0; -2 6 1; 0 1 0];
x = [3; -1; 2];
```

and the equation $\mathbf{y} = \mathbf{A}\mathbf{x}$ is computed as follows

```
y = A*x
```

to produce

```
y =
    3
  -10
   -1
```

Using the $\mathbf{A}$ matrix given above, compute the value of $\mathbf{y}$ in the equation $\mathbf{y} = \mathbf{A}\mathbf{x}$, for the cases:

20. $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

21. $\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

22. $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

23. $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

24. In general, how does the value of of $\mathbf{A}\mathbf{x}$ compare to the value of $\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ or $\mathbf{x}^{\mathrm{T}}\mathbf{A}$?