Homework 4: Reference Frames and Transformations

Assigned	Mon. 13-Feb-2017
Opportunities to Ask Questions	Mon. 13-Feb-2017 (during and after class)
	Wed. 15-Feb-2017 (after class)
	Fri. 17-Feb-2017, 3:40 - 4:55 (Pangborn 207 – extra recitation)
	Mon. 20-Feb-2017 (after class)
Due	Wed. 22-Feb-2017 A late policy of -15% per day late will be strictly enforced.

Workout Problems

Complete the following problems using a pencil, paper and calculator (if necessary). Show all of your work. Grading will be based on the correct answer and clarity/detail of the solution process.

1. Consider Fig. 1 that shows two collocated references frames F_A and F_B with θ (measured counter-clockwise from the $\hat{\mathbf{x}}_A$ axis to the $\hat{\mathbf{x}}_B$ axis).

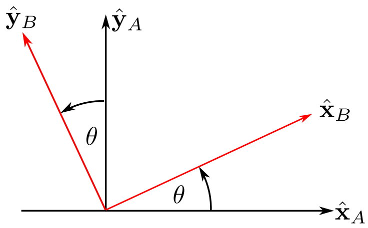

Figure 1: Reference frame F_B is rotated by an angle θ relative to F_A

   (a) Express the unit vectors of the F_B frame ($\hat{\mathbf{x}}_B$ and $\hat{\mathbf{y}}_B$) in the F_B frame (i.e., write the 2D expression for $(\hat{\mathbf{x}}_B)_B$ and $(\hat{\mathbf{y}}_B)_B$).
   (b) Draw a sketch of the $\hat{\mathbf{x}}_B$ unit vector and the F_A unit vectors (only). Indicate the the angle θ. Express the $\hat{\mathbf{x}}_B$ vector in the F_A frame (i.e., find $(\hat{\mathbf{x}}_B)_A$). *Do not use a rotation matrix.* Instead, find the desired unit vectors using either the dot product or trigonometry.
   (c) Repeat part (b) for the $\hat{\mathbf{y}}_B$ unit vector.
   (d) Write the general expression for a rotation matrix that converts vectors from F_B to F_A.
   (e) How are the vectors you found in parts (b) and (c) related to the rotation matrix in part (d)?

2. Consider Fig. 2 that shows two collocated references frames F_A and F_B inclined at an angle θ (measured counter-clockwise from the $\hat{\mathbf{x}}_A$ axis to the $\hat{\mathbf{x}}_B$ axis) and two vector $\mathbf{r}$ and $\mathbf{s}$.

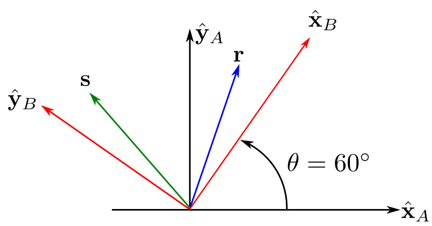

Figure 2: The vectors $\mathbf{r}$ and $\mathbf{s}$ can be expressed in either F_A or F_B. Unit vectors are not drawn to scale.

   (a) Suppose that $(\mathbf{r})_A = (1 \quad 3)^{\mathrm{T}}$.
      i. Sketch the projection of $\mathbf{r}$ onto the F_A axes and label the points $(r_x)_A$ and $(r_y)_A$.
      ii. Sketch the projection of $\mathbf{r}$ onto the F_B axes and label the points $(r_x)_B$ and $(r_y)_B$.
      iii. Find $(\mathbf{r})_B$ using a rotation matrix. Show all your work.
   (b) Suppose that $(\mathbf{s})_B = (1 \quad 3)^{\mathrm{T}}$.
      i. Sketch the projection of $\mathbf{s}$ onto the F_A axes and label the points $(s_x)_A$ and $(s_y)_A$.
      ii. Sketch the projection of $\mathbf{s}$ onto the F_B axes and label the points $(s_x)_B$ and $(s_y)_B$.
      iii. Find $(\mathbf{s})_A$ using a rotation matrix. Show all your work.

3. Consider the problem in Fig. 3. The position of the origin of the camera frame F_C, expressed in the F_U frame, is $(\mathbf{p}_C)_U = (4 \quad 2)^{\mathrm{T}}$. Assume that the angle, measured counter-clockwise, from the $\hat{\mathbf{x}}_U$ axis to the $\hat{\mathbf{x}}_C$ axis is $-15°$. Suppose that a machine vision algorithm returns the position of the quadcopter relative to the camera, expressed in the local F_C frame, as $(\mathbf{r}_{cq})_C = (2.5 \quad 0.5)^{\mathrm{T}}$.

   (a) Find the rotation matrix $\mathbf{R}_C^U$.

   (b) Express the vector $\mathbf{r}_{cq}$ in the F_U frame.

   (c) Give the location of the quadcopter measured from the origin of the F_U frame, and expressed in the F_U frame.

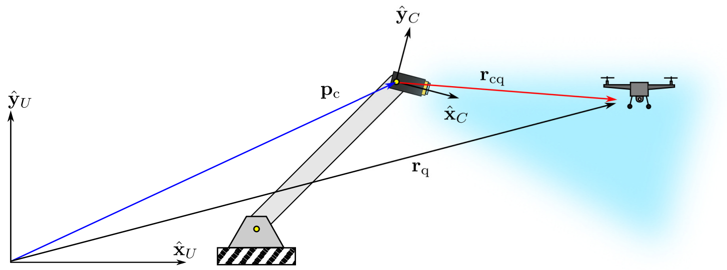

Figure 3: A robotic camera arm detects a quadcopter

4. Consider the two-link robotic manipulator arm shown in Fig. 4. F_U is a non-rotating frame fixed to the base of the manipulator. F_A does not translate but rotates about point A and is aligned with the axis of the first link AB. The length of the link AB is $l_{AB} = 5.0$. The F_B frame translates with point B and rotates about point B is aligned with the axis of the second link BC. The length of the link BC is $l_{BC} = 3.5$.

   (a) Write the homogeneous coordinate $(\mathbf{r}_C)_B$ expressing the position of point C measured from the F_B frame.

   (b) Write the homogeneous coordinate $(\mathbf{r}_B)_A$ expressing the position of point B measured from the F_A frame.

   (c) Find the transformation matrix $\mathbf{T}_B^A$. Sketch the two reference frames.

   (d) Express the position of point C in the F_A frame (i.e., find $(\mathbf{r}_C)_A$).

   (e) Find the transformation matrix $\mathbf{T}_A^U$. Sketch the two reference frames.

   (f) Express the position of point C in the F_A frame (i.e., find $(\mathbf{r}_C)_U$) by using the results from part (d) and (e).

   (g) Find the compound transformation matrix $\mathbf{T}_B^U$.

   (h) Find $(\mathbf{r}_C)_U$ using your result the results from part (a) and (g). Verify that it is the same as your answer in part (f).

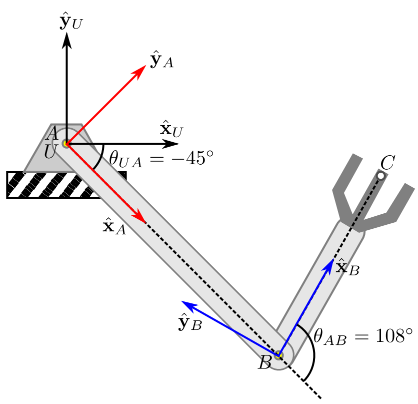

Figure 4: A two-link manipulator

MATLAB Programming Problem

1. Write a MATLAB function of the form

```
T_BA = transformationMatrix(p_AB_aframe, th_AB)
```

   that returns the transformation matrix $\mathbf{T}_B^A$ as a MATLAB matrix `T_BA` given:

   - `p_AB_aframe`: the homogeneous coordinate that points from the origin of the F_A frame to the origin of F_B, expressed in F_A frame
   - `th_AB`: the angle of the F_B frame relative to the F_A frame (measured in radians counter-clockwise from the $\hat{\mathbf{x}}_A$ axis to the $\hat{\mathbf{x}}_B$ axis)

   Submit the following:

   (a) a hardcopy of the `transformationMatrix.m` file

   (b) An m-file script named `main_part1.m` (and corresponding output) that uses the `transformationMatrix.m` function to compute the transformation matrix for the following inputs:

```
p_AB_aframe = [5 -1 1]';
th_AB = 35*pi/180;
```

2. Consider the three-link robotic manipulator shown in Fig. 5. The lengths of the three links are given as

```
Lab = 5; % the black link
Lbc = 2; % the magenta link
Lcd = 3; % the cyan link
```

   Use the function you created in Part 1 of this MATLAB problem to complete the following.

   (a) Submit a hardcopy of you m-file script named `main_part2.m` (and corresponding output) that carries out the computations described below.

   (b) Compute `T_CB` (i.e., the transformation matrix $\mathbf{T}_C^B$)

   (c) Compute `T_BA` (i.e., the transformation matrix $\mathbf{T}_B^A$)

   (d) Compute `T_AU` (i.e., the transformation matrix $\mathbf{T}_A^U$)

   (e) State the position of pt. D expressed in the F_C frame (i.e. $(\mathbf{r}_D)_C$) in homogenous coordinates.

   (f) Compute `T_CU` (i.e., the compound transformation matrix $\mathbf{T}_C^U$)

   (g) Use your answers from (e) and (f) to compute the position of the point D in the F_U frame

3. Suppose that a manipulation task requires the endpoint (point D) to reach a goal region defined as a circle around the point (0,5) with a radius of 0.05 units. We wish to determine the set of joint angles that can reach the goal region. To reduce the computational complexity we only consider joint angles in increments of 5° degrees. Thus, each of the three joints can take one of the 72 values: $\{0°, 5°, 10°, \cdots, 355°\}$.

   (a) Submit a hardcopy of a script named `main_part3.m` (and corresponding output) that solves the above problem.

   (b) What is the total number of joint angle combinations you must consider?

   (c) Enumerate the N solutions that reach the goal region. Format this output as a matrix with N rows (one for each solution) and three columns to indicate the joint angles of the first (black), second (magenta), and third (cyan) link respectively (in degrees).

   (d) From the list you generated in (c) submit three (3) plots of unique configurations that reach the goal region. Generate the plots using `drawThreeLinkArm.p`. Label the joint angles corresponding to each plot.

4. BONUS. (Optional question worth extra points): The owner of the manipulator arm wants to optimize production and determine which set of joint angles reaches the goal region the fastest from a start position with the the manipulator completely horizontal (all angles 0°). If we assume that all joints move simultaneously, at a fixed rate, then the speed with which the task is accomplished is proportional to the largest change in joint angle. Give the joint angle triplet from the set determined in Part 3 that reaches the goal region the fastest. Give the joint angle triplet from the set determined in Part 3 that reaches the goal region the slowest. Make sure you consider angle wrap-around (i.e., a joint angle of 300° has the same angular distance, measured from 0° as a joint angle of 30°). What is the ratio of the fastest to slowest configuration?

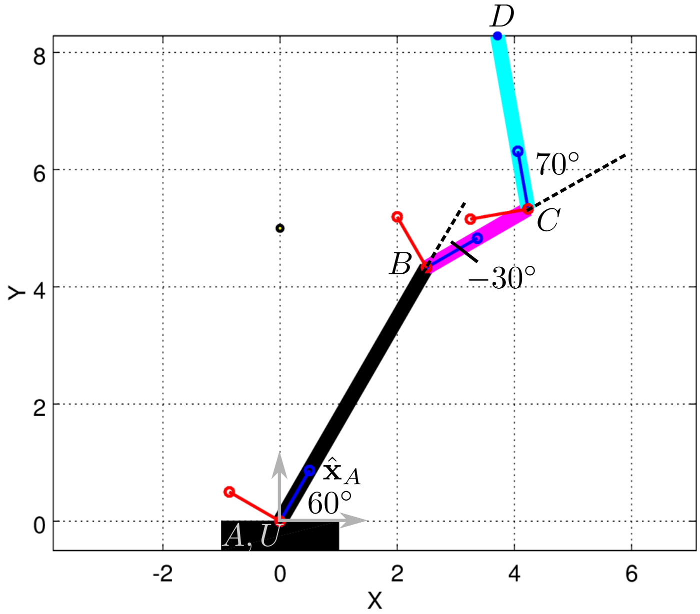

Figure 5: Robotic manipulator with three links

Hints:

1. You can verify your code by using the `drawThreeLinkArm.p` script provided. The data cursor can be used to give the locations of link endpoints on the resulting figure. The syntax for using `drawThreeLinkArm` is:

```
% Usage: drawThreeLinkArm(th_AU, th_BA, th_CB)
%
% Produces a simple drawing of a three-link arm given the
% the lengths of each link and *relative* joint angles. There are three links
% labelled A, B, and C. The A link begins at the origin of the universal
% reference frame F_U.
%
% Inputs:
% th_AU : (degrees) angle of F_A relative to F_U
% th_BA : (degrees) angle of F_B relative to F_A
% th_CB : (degrees) angle of F_C relative to F_B
%
% Outputs:
% a plot of the three link arm and the goal region
```