Homework 7: Probability and Error Propagation

Assigned: Mon. 20-March
Due: Wed. 29-March

1. (3 points total) By looking at Fig. 2 in the notes (Section 7) sketch (by hand) the following probability density functions corresponding to the three random variables X, Y, and Z on a single set of axes:

   - $X \sim \mathcal{N}(0, 2^2)$
   - $Y \sim \mathcal{N}(2, 2^2)$
   - $Z \sim \mathcal{N}(1, 0.8^2)$.

   Make sure you label each curve, label the axes, and include ticks/numbers to indicate the scale.

2. (6 points total) For each of the following statements, identify the mean μ and standard deviation σ.

   - The random variable X is between 2.0 and 3.0 about 95 % of the time.
   - The random variable X is between 1.0 and 4.0 about 68 % of the time.
   - The random variable X is between -8.0 and -2.0 about 99.7 % of the time.

3. (6 points total) Given the random variable $X \sim \mathcal{N}(5, 0.1)$, compute the mean μ_y and variance σ_y^2 of the variable Y given by the following transformations (use the 1D error propagation law given in the second last example of Section 7):

   - $y = f(x) = x^2$
   - $y = f(x) = 4x + 1$
   - $y = f(x) = x\cos(x)$

4. (6 points total) Modify `plot_random2DVectorPDF.m` and plot the contours of the probability density functions corresponding to the three random vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$:

   - $\mathbf{x} \sim \mathcal{N}\left(\boldsymbol{\mu}_\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{P}_\mathbf{x} \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}\right)$
   - $\mathbf{x} \sim \mathcal{N}\left(\boldsymbol{\mu}_\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{P}_\mathbf{x} \begin{pmatrix} 3 & -3 \\ -3 & 10 \end{pmatrix}\right)$
   - $\mathbf{x} \sim \mathcal{N}\left(\boldsymbol{\mu}_\mathbf{x} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \mathbf{P}_\mathbf{x} \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}\right)$

   Submit a printout of each distribution (total of three plots).

5. (20 points total) Suppose that an autonomous car is equipped with a sonar sensor that measures both range x_1 and bearing x_2 (measured CCW from the horizontal) to nearby obstacles (e.g., a traffic sign, pedestrians, other vehicles). The sensor is located at the origin so that the z_1-z_2 coordinate of an obstacle measured at a given range and bearing is:

   $$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos x_2 \\ x_1 \sin x_2 \end{pmatrix} \tag{1}$$

   Suppose that at the particular instant shown in Fig. 1 the sensor reports an obstacle at a range of 150 m and at a bearing $30\pi/180$ rad. We assume that there is some uncertainty in this measurement.

Once approach is to treat these measurements as statistically independent random variables $X_1 \sim \mathcal{N}(\mu_{x_1}, \sigma_{x_1}^2)$ with $\mu_{x_1} = 150$ m and $\sigma_{x_1} = 1$ m, and $X_2 \sim \mathcal{N}(\mu_{x_2}, \sigma_{x_2}^2)$ where $\mu_{x_2} = 30\pi/180$ rad. and $\sigma_{x_2} = 5\pi/180$ rad.

However, since we have a vector-valued output $\mathbf{z}$ defined by the function (1) we should instead treat these two measurements as a *random vector*. Let $\mathbf{x} = (x_1 \;\; x_2)^{\mathrm{T}}$ and $\mathbf{z} = (z_1 \;\; z_2)^{\mathrm{T}}$ be random vectors related by the system (1).

Make sure to use radians throughout this problem instead of degrees. (Using degrees can lead to derivative errors if not accounted for.)

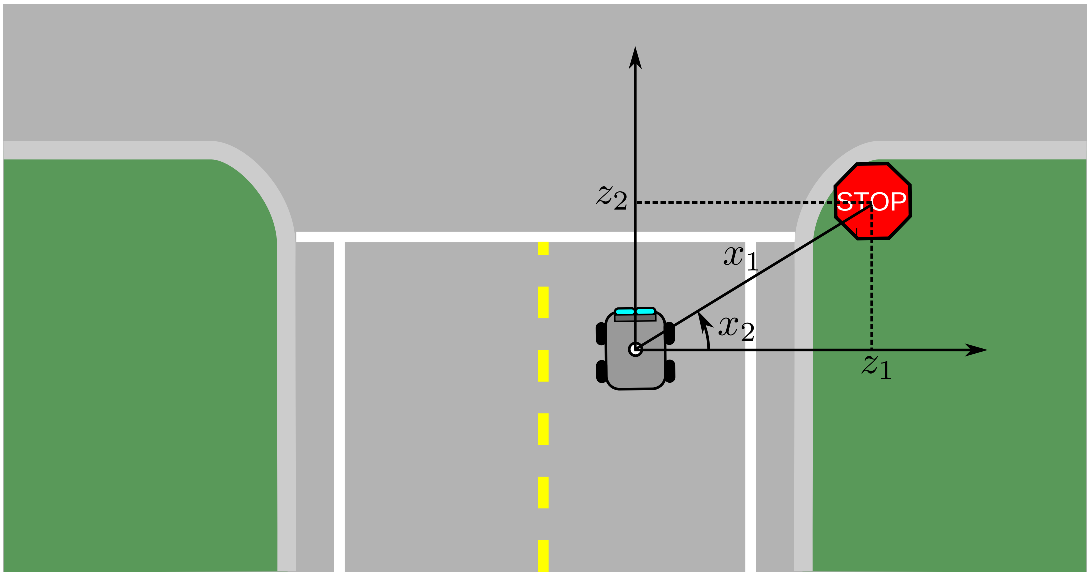

Figure 1: An autonomous car measuring the range and bearing to a nearby stop sign.

(a) (1 points) State the mean $\boldsymbol{\mu}_{\mathbf{x}}$ based on the above paragraph.

(b) (2 points) State the corresponding covariance matrix $\mathbf{P}_{\mathbf{x}}$ given that x_1 and x_2 are uncorrelated and have variance as described above. (Make sure to convert from standard deviation to variance!)

(c) (2 points) Compute the mean $\boldsymbol{\mu}_{\mathbf{z}}$.

(d) (3 points) Compute the Jacobian $\mathbf{J}_{\mathbf{f}}$ of the system (1) symbolically.

(e) (2 points) Evaluate the Jacobian with the mean $\boldsymbol{\mu}_{\mathbf{x}}$.

(f) (2 points) Use the error propagation law to determine the covariance matrix $\mathbf{P}_{\mathbf{z}}$.

(g) (3 points) Simulate $N = 500$ random measurements from the distributions $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{x}}, \mathbf{P}_{\mathbf{x}})$ you determined above (use the `randSamplesWithMeanCov.m` function provided). Plot these samples with the x_1 values on the x-axis and the x_2 values on the y-axis. Use circular markers without a connecting line. Use a large legible font size and label axes.

(h) (3 points) Propagate each sample from (g) using the system (1). Plot the resulting samples with the z_1 values on the x-axis and the z_2 values on the y-axis. Use circular markers without a connecting line. Use a large legible font size and label axes.

(i) (2 points) Compute and mean the covariance of the samples in (h) (using the `meanCovarianceFromSamples.m` function provided). How does this compare to your result in (f)?