Section 4: Reference Frames and Transformations

Much of robotics deals with manipulating objects or moving through space. Therefore it is necessary that we develop the tools necessary to describe positions and rotations. A *reference frame* is a coordinate system in which a particular quantity is expressed. Reference frames can be fixed to some point in space, or they can be moving and rotating.

The Universal Reference Frame F_U

We assume that every robotic system always has some *universal reference frame* that is not moving or rotating, we call this reference frame F_U. For example, the position of an object in this reference frame can be denoted by a vector such as

$$(\mathbf{r})_U = \begin{pmatrix} (r_x)_U \\ (r_y)_U \\ (r_z)_U \end{pmatrix}$$

In this course, we will use rounded brackets proceeded by a subscript to denote the reference frame in which a vector is expressed. In the above example $(\mathbf{r})_U$ indicates that the vector $\mathbf{r}$ is numerically expressed in the F_U frame. The moment we assign specific values to $\mathbf{r}$ it is implicitly assumed that we have chosen a particular reference frame. Since we will be dealing with many different coordinate frames this notation will help clarify in which frame the vector is expressed. Although we mention a position vector above, numerous other vector quantities are used in robotics (e.g., to represent forces, torques, velocities).

Numerically Expressing Vectors

Consider a two-dimensional vector $\mathbf{r}$ and the three reference frames in Fig. 1. These three frames might represent, for example, snapshots of a single frame that is rigidly attached to a rotating object. The vector $\mathbf{r}$ does not change in each case, but in each reference frame $\mathbf{r}$ is represented by a different set of numeric values. For example, in the frame F_A the vector $\mathbf{r}$ is numerically represented by

$$(\mathbf{r})_A = \begin{pmatrix} (r_x)_A \\ (r_y)_A \end{pmatrix}$$

The number $(r_x)_A$ represents the projection of the vector $\mathbf{r}_A$ onto the $\hat{\mathbf{x}}_A$ axis. Similarly, the numeric values of $\mathbf{r}_B$ and $\mathbf{r}_C$ will be different since their axes are different.

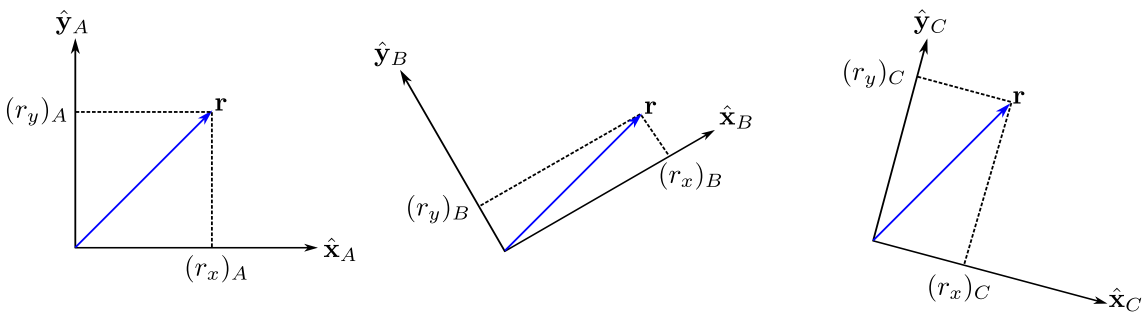


Figure 1: The vector $\mathbf{r}$ in three reference frames: F_A, F_B and F_C. Note that the numeric representation of the vector will be different in each frame, even though the vector remains unchanged.

Side Note: Projecting Vectors and the Dot Product

For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, the projection of $\mathbf{v}_1$ onto $\mathbf{v}_2$ (or vice versa) is the dot product

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = ||v_1||\;||v_2|| \cos\theta \tag{1}$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$. In the example above, the value $(r_x)_A$ represents the projection of the $\mathbf{r}$ vector onto the $\hat{\mathbf{x}}_A$ unit vector, in other words: $(r_x)_A = \mathbf{r} \cdot \hat{\mathbf{x}}_A$

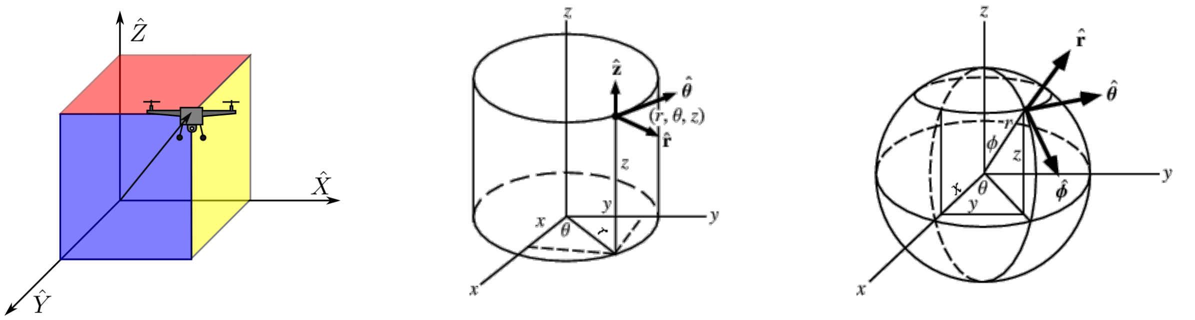
Common Reference Frames

A *Cartesian* reference frame F_A consists of the unit vectors $\hat{\mathbf{x}}_A$, $\hat{\mathbf{y}}_A$ and $\hat{\mathbf{z}}_A$ corresponding to three perpendicular planes (see Fig. 2a). In the F_A frame itself these are all vectors with all zeros entries and one entry equal to one, in three dimensions:

$$(\hat{\mathbf{x}}_A)_A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (\hat{\mathbf{y}}_A)_A = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad (\hat{\mathbf{z}}_A)_A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

However, these unit vectors can also be expressed in a different reference frame. For example, if there is some other reference frame F_B, these unit vectors will take on other numeric values and will be denoted $(\hat{\mathbf{x}}_A)_B$, $(\hat{\mathbf{y}}_A)_B$ and $(\hat{\mathbf{z}}_A)_B$. (We will return to this point later.)

In this course we will mainly use a Cartesian reference frame, but other non-Cartesian coordinate systems are also possible: such as *spherical* or *cylindrical* coordinates (Figs. 2b and 2c). The choice of reference frame depends on the problem at hand; although the Cartesian frame is the natural first choice, sometimes using a spherical or cylindrical system can make the problem much simpler.



(a) Cartesian reference frame

(b) Cylindrical reference frame (Courtesy of Wolfram Mathworld)

(c) Spherical reference frame (Courtesy of Wolfram Mathworld)

Figure 2: Common reference frames

Side Note: Vector and Matrix Notation

In this course we will use the following convections:

- plain text denotes scalars (e.g., v)
- lower case bold type denotes vectors (e.g., $\mathbf{v}$)
- unit vectors will be denoted with a ˆ symbol (e.g., $\hat{\mathbf{x}}$)

- the transpose of a vector is denoted with a superscript $^{\mathrm{T}}$ symbol

  e.g., if $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ then $\mathbf{v} = (v_1 \; v_2 \; \cdots \; v_n)^{\mathrm{T}}$ and $\mathbf{v}^{\mathrm{T}} = (v_1 \; v_2 \; \cdots \; v_n)$

- upper case bold type denotes matrices (e.g., $\mathbf{V}$)
- when we say a matrix $\mathbf{A}$ is an $m \times n$ matrix it implies that $\mathbf{A}$ has m rows and n columns
- the matrix transpose is denoted with the superscript $^{\mathrm{T}}$ symbol

  e.g., if $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

- a square identity matrix with n rows and n columns is denoted by $\mathbf{1}_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}$
- multiplying a vector or matrix by the identity matrix leaves the vector or matrix unchanged

  e.g. $\mathbf{1}_{n\times n}\mathbf{v} = \mathbf{v}$ or $\mathbf{1}_{n\times n}\mathbf{A} = \mathbf{A}$

- the inverse of a matrix is denoted $\mathbf{A}^{-1}$
- Fact: if $\mathbf{A}$ is a square $n \times n$ matrix then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{1}_{n\times n}$ and the matrices "cancel"
- An example equation that illustrates some of this notation is the eigenvalue equation: $\lambda\mathbf{v} = \mathbf{A}\mathbf{v}$ which includes a scalar eigenvalue λ, an eigenvector $\mathbf{v}$, and a matrix $\mathbf{A}$.

A Motivating Problem

Consider the robotic arm shown in Fig. 3. A rotating camera is attached at the end of the arm. The vector $\mathbf{p}_{\mathrm{c}}$ describes where the camera is located with respect to the origin of the F_U frame. Suppose that

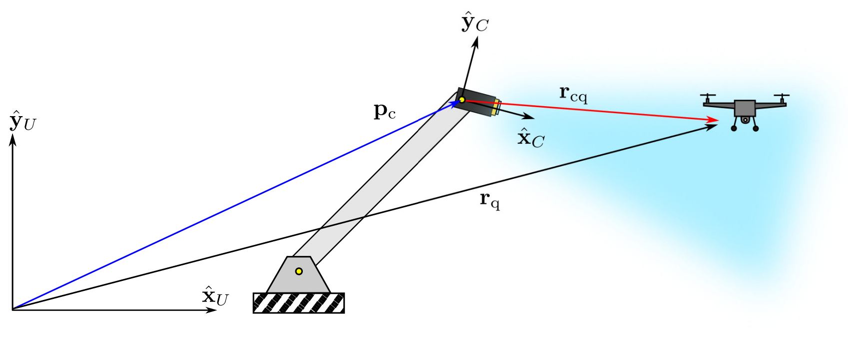


Figure 3: The position and orientation of a planar robotic arm

the position of quadcopter relative to the camera is $\mathbf{r}_{\mathrm{cq}}$. Thus the position of the quadcopter relative to the origin of the F_U frame is

$$\mathbf{r}_q = \mathbf{p}_{\mathrm{c}} + \mathbf{r}_{\mathrm{cq}} \tag{3}$$

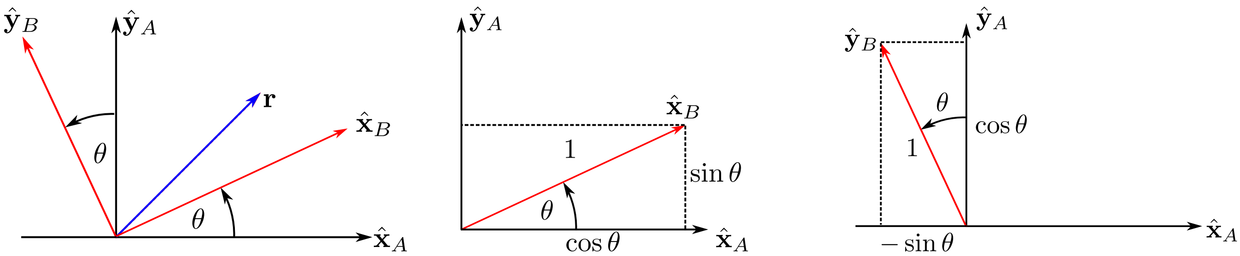
To carry out the computation, and obtain numeric values for $\mathbf{r}_q$, we must choose a consistent reference frame in which to express each of the three vectors in Eq. 3. A natural choice is to use the F_U reference frame. Thus we can re-write Eq. 3 as

$$(\mathbf{r}_q)_U = (\mathbf{p}_c)_U + (\mathbf{r}_{cq})_U \tag{4}$$

to emphasize that all of the vectors being added are expressed in the F_U frame. Suppose we already have $\mathbf{p}_c$ in the F_U frame (i.e., we have $(\mathbf{p}_c)_U$), but the vector $\mathbf{r}_{cq}$ is given in the F_C frame as $(\mathbf{r}_{cq})_A$. This is a very common situation. For example, the F_C frame might represent the direction in which the camera is pointed and a machine vision algorithm that uses the camera images might output relative position vectors of objects it detects in this frame only. It is clear that we need a way to convert a vector in the frame F_C to a vector in the frame F_U so that we can carry out computations such as the one in Eq. 4. As we will see in the following section, the enabling tool is a *rotation matrix*.

2D Rotation Matrices

Besides knowing the position of an object we are often interested in the orientation of it. A common way to describe the orientation of a rigid body is to attach a fixed reference frame to it. This body-attached reference frame is not necessarily used to express any other vectors – *the reference frame itself expresses the orientation of the rigid body.* One way to relate two reference frames, F_A and F_B, is to express the unit vectors of the F_B reference frame using the F_A frame (i.e., to express $(\hat{\mathbf{x}}_B)_A$, $(\hat{\mathbf{y}}_B)_A$ and $(\hat{\mathbf{z}}_B)_A$). Then by assembling these vectors as columns of a new matrix we obtain a rotation matrix $\mathbf{R}_B^A$ which can be used to convert vectors from one frame to the other, and vice versa. The following example will illustrate this concept for a two-dimensional reference frame. Consider the two reference frames in



(a) The reference frame F_A is rotated by an angle θ to produce F_B

(b) Expressing the $\hat{\mathbf{x}}_B$ unit vector in the F_A frame as $(\hat{\mathbf{x}}_B)^A$

(c) Expressing the $\hat{\mathbf{y}}_B$ unit vector in the F_A frame as $(\hat{\mathbf{y}}_B)^A$

Figure 4: A 2D passive rotation.

Fig. 4a. The vector $\mathbf{r}$ can be represented as either $(\mathbf{r})_A$ or $(\mathbf{r})_B$. The unit vectors $\hat{\mathbf{x}}_B$ and $\hat{\mathbf{y}}_B$ have a length of one (by definition) thus $||(\hat{\mathbf{x}}_B)_A|| = ||(\hat{\mathbf{y}}_B)_A|| = 1$. To express the unit vectors of the F_B frame in the F_A frame we need to find the projection of each unit vector onto the $\hat{\mathbf{x}}_A$ and $\hat{\mathbf{y}}_A$ axes. Using Eq. 1 we have:

$$\begin{aligned}
\hat{\mathbf{x}}_B \cdot \hat{\mathbf{x}}_A &= \cos\theta \\
\hat{\mathbf{x}}_B \cdot \hat{\mathbf{y}}_A &= \cos(\pi/2 - \theta) = \sin\theta \\
\hat{\mathbf{y}}_B \cdot \hat{\mathbf{x}}_A &= \cos(\pi/2 + \theta) = -\sin\theta \\
\hat{\mathbf{y}}_B \cdot \hat{\mathbf{y}}_A &= \cos\theta
\end{aligned}$$

where the angle argument of the cosine was determined by inspecting Fig. 4 for each case. Alternately, we could have arrived at the same results using trigonometry.

Regardless of the approach used, we obtain

$$(\hat{\mathbf{x}}_B)_A = \begin{pmatrix} \hat{\mathbf{x}}_B \cdot \hat{\mathbf{x}}_A \\ \hat{\mathbf{y}}_B \cdot \hat{\mathbf{y}}_A \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad (\hat{\mathbf{y}}_B)_A = \begin{pmatrix} \hat{\mathbf{y}}_B \cdot \hat{\mathbf{x}}_A \\ \hat{\mathbf{y}}_B \cdot \hat{\mathbf{y}}_A \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{5}$$

The rotation matrix can be constructed by using the vectors in (5) as the columns of the new matrix:

$$\mathbf{R}_B^A(\theta) = ((\hat{\mathbf{x}}_B)_A \quad (\hat{\mathbf{y}}_B)_A) = \begin{pmatrix} \hat{\mathbf{x}}_B \cdot \hat{\mathbf{x}}_A & \hat{\mathbf{y}}_B \cdot \hat{\mathbf{x}}_A \\ \hat{\mathbf{x}}_B \cdot \hat{\mathbf{y}}_A & \hat{\mathbf{y}}_B \cdot \hat{\mathbf{y}}_A \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

When we multiply a vector $(\mathbf{v})_B$ by this matrix we obtain the desired result:

$$(\mathbf{v})_A = \mathbf{R}_B^A(\theta)(\mathbf{v})_B \tag{7}$$

The rotation matrix $\mathbf{R}_B^A$ has subscript B and superscript A to denote that it converts vectors from the frame F_B to F_A. To convince yourself that Eq. 7 indeed converts vectors as claimed, consider what happens when we pre-multiply the unit vectors $\hat{\mathbf{x}}_B$ and $\hat{\mathbf{y}}_B$, expressed in the F_B frame, by the rotation matrix – we obtain:

$$\mathbf{R}_B^A(\theta)(\hat{\mathbf{x}}_B)_B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = (\hat{\mathbf{x}}_B)_A$$

and

$$\mathbf{R}_B^A(\theta)(\hat{\mathbf{y}}_B)_B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = (\hat{\mathbf{y}}_B)_A$$

which are exactly the expressions in Eq. 5. Since any vector in the F_B frame is simply a linear combination of the unit vectors then clearly the same procedure applies.

The matrix (6) is parametrized by the angle θ thus we can use it as a general formula for any rotation of 2D reference frames. Note that θ is measured counter-clockwise from the $\hat{\mathbf{x}}_A$ axis to the $\hat{\mathbf{x}}_B$ axis which is, in general, *not from the horizontal*. (In the following we drop the reference to θ and assume that this is implied from context.) However, Eq. 7 can only convert vectors one way: from the F_B frame to the F_A frame. If Eq. 7 was a scalar equation we could obtain $\mathbf{v}_B$ by diving boths sides by $\mathbf{R}_B^A$. But recall that matrix division is not possible – instead we must "invert" the matrix to obtain $(\mathbf{R}_B^A)^{-1}$ and pre-multiply both sides. The quantity $(\mathbf{R}_B^A)^{-1}(\mathbf{R}_B^A) = \mathbf{1}_{n\times n}$ is equal to the identity matrix (a n-by-n diagonal matrix of ones) and effectively "cancels" itself. Thus

$$\begin{aligned} (\mathbf{v})_A &= (\mathbf{R}_B^A)(\mathbf{v})_B \\ (\mathbf{R}_B^A)^{-1}(\mathbf{v})_A &= (\mathbf{R}_B^A)^{-1}(\mathbf{R}_A^B)(\mathbf{v})_B \\ (\mathbf{R}_B^A)^{-1}(\mathbf{v})_A &= \mathbf{1}_{n\times n}(\mathbf{v})_B \\ (\mathbf{R}_B^A)^{-1}(\mathbf{v})_A &= (\mathbf{v})_B \end{aligned}$$

Since $(\mathbf{R}_B^A)^{-1}$ converts vectors from the F_A frame to the F_B frame we denote it as $\mathbf{R}_A^B(\theta)$

$$\mathbf{R}_A^B = (\mathbf{R}_B^A)^{-1}$$

Thus we have the desired conversion, but it requires that we compute the matrix inverse $(\mathbf{R}_A^B)^{-1}$. For generic matrices this can be computed numerically but it is cumbersome to compute analytically. Alternately, we could obtain $\mathbf{R}_B^A$ by repeating the same procedure we used to arrive at $\mathbf{R}_A^B$. This would involve expressing the F_B frame unit vectors in the F_A frame. Upon closer inspection of Eq. 6 it turns out that we've already done this! Notice that the rows of Eq. 6 are the vectors

$$[(\hat{\mathbf{x}}_A)_B]^{\mathrm{T}} = \begin{pmatrix} \hat{\mathbf{x}}_A \cdot \hat{\mathbf{x}}_B & \hat{\mathbf{x}}_A \cdot \hat{\mathbf{y}}_B \end{pmatrix} \qquad \text{and} \qquad [(\hat{\mathbf{y}}_A)_B]^{\mathrm{T}} = \begin{pmatrix} \hat{\mathbf{y}}_A \cdot \hat{\mathbf{x}}_B & \hat{\mathbf{y}}_A \cdot \hat{\mathbf{y}}_B \end{pmatrix}$$

Therefore we can obtain $\mathbf{R}_A^B$ from $\mathbf{R}_B^A$ by simply transposing the rows into columns. In other words:

$$\boxed{\mathbf{R}_A^B = (\mathbf{R}_B^A)^{-1} = (\mathbf{R}_B^A)^{\mathrm{T}}}$$

Note that the way θ is defined is the same in $\mathbf{R}_A^B$ and $\mathbf{R}_B^A$. Be careful to clearly identify, for a given problem, what reference frame corresponds to F_A, what reference corresponds to F_B, and how θ is measured. Rotation matrices are a special case in which the transpose is equal to the inverse of the matrix (i.e., they are *orthogonal* matrices). To summarize, if frame F_B is rotated by an angle of θ relative to F_A we have the following two equations that convert vectors from one frame to another:

$$(\mathbf{v})_A = \mathbf{R}_B^A(\mathbf{v})_B = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (\mathbf{v})_B \tag{8}$$

$$(\mathbf{v})_B = \mathbf{R}_A^B(\mathbf{v})_A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} (\mathbf{v})_A \tag{9}$$

In Eqs. (8)-(9) the angle θ is defined counter-clockwise from the $\hat{\mathbf{x}}_A$ axis to the $\mathbf{x}_B$ axis.

Example: 2D Rotation Matrices from F_A to F_B

Suppose that the vector $\mathbf{r}$ given in the F_A frame (Fig. 5) is

$$(\mathbf{r})_A = \begin{pmatrix} (r_x)_A \\ (r_y)_A \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

Now suppose the frame F_B is at an angle relative to F_A (measured counter-clockwise from the $\hat{\mathbf{x}}_A$ axis

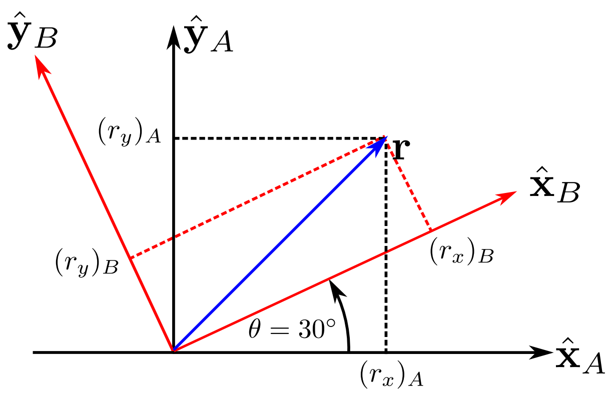


Figure 5: Rotating the $\mathbf{r}$ vector from F_A to F_B

to the $\hat{\mathbf{x}}_B$ axis) of $\theta = 30°$ relative to F_A. Then the rotation matrix that converts vectors from the F_A frame to the F_B frame can be evaluated as

$$\mathbf{R}_A^B\big|_{\theta=30°} = \begin{pmatrix} \cos(30°) & \sin(30°) \\ -\sin(30°) & \cos(30°) \end{pmatrix} = \begin{pmatrix} 0.8660 & 0.5000 \\ -0.5000 & 0.8660 \end{pmatrix}$$

The vector $\mathbf{r}$ in the F_B frame is then computed using Eq. 9

$$\begin{aligned}(\mathbf{r})_B &= \mathbf{R}_A^B(\mathbf{r})_A \\ &= \begin{pmatrix} 0.8660 & 0.5000 \\ -0.5000 & 0.8660 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4.098 \\ 1.098 \end{pmatrix}\end{aligned}$$

The fact that $(r_x)_B > (r_y)_A$ is consistent with Fig. 1 where it is clear that $\mathbf{r}_B$ has a large component projected onto the unit vector $\hat{\mathbf{x}}_B$.

Example: 2D Rotation Matrices from F_B to F_A
Suppose that the vector $\mathbf{r}$ given in the F_B frame (Fig. 5) is

$$(\mathbf{r})_B = \begin{pmatrix} (r_x)_B \\ (r_y)_B \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

Now suppose the frame F_B is at an angle relative to F_A (measured counter-clockwise from the $\hat{\mathbf{x}}_A$ axis

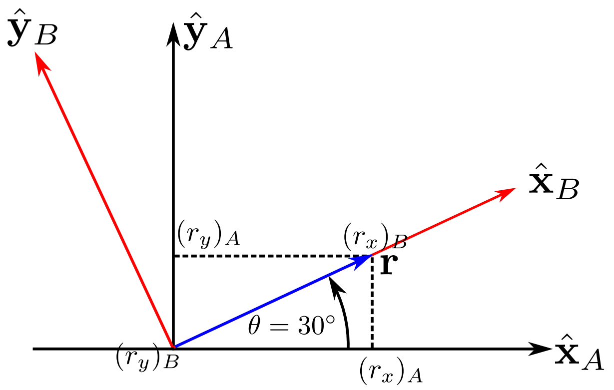


Figure 6: Rotating the $\mathbf{r}$ vector from F_B to F_A

to the $\hat{\mathbf{x}}_B$ axis) of $\theta = 30°$ relative to F_A. Then the rotation matrix that converts vectors from the F_B frame to the F_A frame can be evaluated as

$$\mathbf{R}_B^A\big|_{\theta=30°} = \begin{pmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{pmatrix} = \begin{pmatrix} 0.8660 & -0.5000 \\ 0.5000 & 0.8660 \end{pmatrix}$$

The vector $\mathbf{r}$ in the F_B frame is then computed using Eq. 9

$$\begin{aligned}(\mathbf{r})_B &= \mathbf{R}_A^B(\mathbf{r})_A \\ &= \begin{pmatrix} 0.8660 & -0.5000 \\ 0.5000 & 0.8660 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0.433 \\ 0.250 \end{pmatrix}\end{aligned}$$

Reference frames won't always appear with the F_A axes pointing in the familiar left and up directions. Consider Fig. 7, in which the $\hat{\mathbf{x}}_A$ axis points up and the $\hat{\mathbf{y}}_A$ axis points to the left. The angle θ used in defining the rotation matrix is measured from $\hat{\mathbf{x}}_A$ axis to the $\hat{\mathbf{x}}_B$ axis – in this case that means the angle can be written as negative.

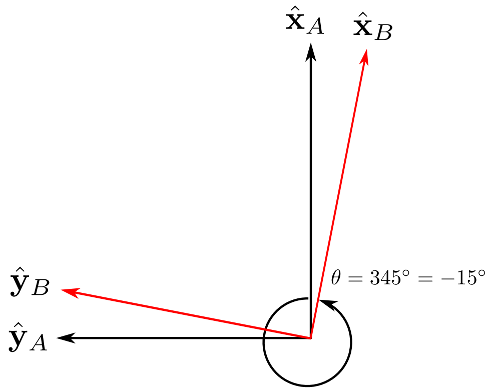


Figure 7: Atypical reference frame orientations

Rigid Transformations and Homogeneous Coordinates

In our discussion of 2D rotation matrices we learned how to express a vector in two different reference frames that were rotated relative to each other. This is applicable to vectors that have no spatial reference point or to convert between vectors that *shared a common origin*. We called each of these representations $(\mathbf{r})_A$ and $(\mathbf{r})_B$. Such *rotations* preserved the length of the vector (i.e., $||(\mathbf{r})_A|| = ||(\mathbf{r})_B||$).

Now we will discuss *rigid transformations* between reference frames. Rigid transformations are used to determine two vectors also called $(\mathbf{r})_A$ and $(\mathbf{r})_B$, each expressed in different reference frames, but that both *point to the same position in space*. It is a slight abuse of notation to call these vectors $(\mathbf{r})_A$ and $(\mathbf{r})_B$ since they are no longer the *same vector* as in the case of the rotation transformations above. Since the reference frame origins are not collocated the two vectors will not have the same length in general (i.e., $|(\mathbf{r})_A| \neq ||(\mathbf{r})_B||$). This situation is illustrated in Fig. 8, both $(\mathbf{r})_A$ and $(\mathbf{r})_B$ point to the same position in space (point Q) but each originates from the origin of a different reference frame.

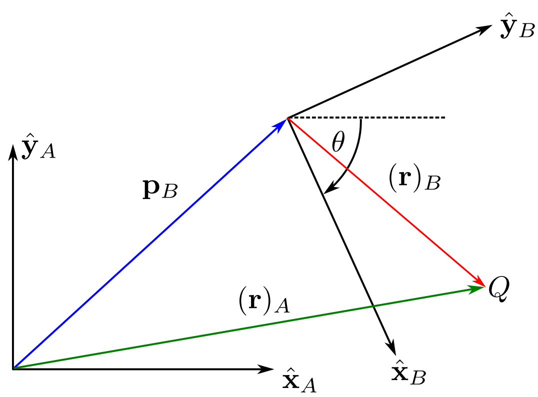


Figure 8: An example of a transformation matrix

Let $(\mathbf{p}_B)_A$ be the position of the origin of the F_B frame relative to the F_A frame, and let $\mathbf{R}_B^A$ be the rotation matrix that describes how F_B is oriented relative to F_A. Then we have the necessary mathematical machinery to formally define a reference frame as a set of two objects:

$$F_B = \{ \ \mathbf{R}_B^A \ , \ (\mathbf{p}_B)_A \ \} \tag{10}$$

In this definition, the F_A frame is not necessarily the universal F_U frame but it must be uniquely defined in relation to the F_U frame. (For example, the F_A frame itself may be defined relative to the F_U frame.) To account for the fact that a position vector measured from F_B has a different start point than one that is measured in F_A we must add the difference when mapping between reference frames to obtain the rigid transformation

$$(\mathbf{r})_A = \underbrace{(\mathbf{R})^A_B(\mathbf{r})_B}_{\text{rotating}} + \underbrace{(\mathbf{p}_B)_A}_{\text{translating}} \tag{11}$$

Although Eq. 11 seems quite simple it can become complicated when we have many intermediate reference frames that we are converting from. A simpler representation is to represent Eq. 11 with a single *transformation matrix*, $\mathbf{T}^A_B$ that accounts for both translation and rotation so that

$$(\mathbf{r})_A = \mathbf{T}^A_B(\mathbf{r})_B \tag{12}$$

However, this convenient expression is only possible if we slightly modify how we express our vectors. In particular, we have to use planar *homogeneous coordinates*. With homogeneous coordinates a point $\mathbf{p} = (p_x \quad p_y)^{\mathrm{T}}$ is represented by the column vector

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} \tag{13}$$

The 3rd entry (equal to one) is what makes this a homogeneous coordinate and what makes it possible to represent translations through matrix multiplication. To avoid special notation we assume that it is apparent from context whether the vector is a homogeneous coordinate. Normally, if there are two planar vectors, $\mathbf{a}$ and $\mathbf{p}$, then we add them as follows

$$\mathbf{a} + \mathbf{p} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a_1 + p_x \\ a_2 + p_y \end{pmatrix} \tag{14}$$

Using homogeneous coordinates, we can "add" the two homogeneous vectors by using matrix multiplication – i.e., there exists a homogeneous translation matrix $\mathbf{T}_{\text{trans}}(\mathbf{p})$ such that

$$\begin{aligned} \mathbf{a} + \mathbf{p} &= \mathbf{T}_{\text{trans}}(\mathbf{p})\mathbf{a} \\ &= \underbrace{\begin{pmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{T}_{\text{trans}}(\mathbf{p})} \begin{pmatrix} a_1 \\ a_2 \\ 1 \end{pmatrix} = \begin{pmatrix} (1)a_1 + (0)a_2 + (p_x)1 \\ (0)a_1 + (1)a_2 + (p_y)1 \\ (0)a_1 + (0)a_2 + (1)1 \end{pmatrix} = \begin{pmatrix} a_1 + p_x \\ a_2 + p_y \\ 1 \end{pmatrix} \end{aligned} \tag{15}$$

Note that Eq. 14 and Eq. 15 produce the same result, except that the result in Eq. 15 is a homogeneous coordinate. Transformation matrices are especially useful because they can combine both rotations *and* translations into one matrix operation and thus they can express rigid vector transformations.

Recall our example in which the reference frame F_B is given in relation to reference frame F_A by $\mathbf{R}^A_B$ and $(\mathbf{p}_B)_A = ((p_{B_x})_A \quad (p_{B_y})_A)^{\mathrm{T}}$. We can convert vectors from the F_B frame to the F_A frame using the following transformation matrix

$$\mathbf{T}^A_B = \begin{pmatrix} \cos\theta & -\sin\theta & (p_{B_x})_A \\ \sin\theta & \cos\theta & (p_{B_y})_A \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

Note that the 2 x 2 upper left matrix is exactly equal to $\mathbf{R}_B^A$.

Example: 2D Transformation Matrices

Refer back to Fig. 8. Suppose that the F_B frame is rotated by an angle $\theta = -60°$ relative to $\hat{\mathbf{x}}_A$. The position of a point Q is expressed in the F_B frame in homogenous coordinates is

$$(\mathbf{r})_B = \begin{pmatrix} (r_x)_B \\ (r_y)_B \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

The origin of the F_B frame relative to the origin of the F_A frame (and expressed in the F_A frame) is

$$(\mathbf{p}_B)_A = \begin{pmatrix} (p_{B_x})_A \\ (p_{B_y})_A \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix}$$

To compute the position of the point Q in the F_A frame we use Eqs. 12 and 16.

$$\begin{aligned}(\mathbf{r})_A &= \mathbf{T}_B^A(\mathbf{r})_B \\ &= \begin{pmatrix} \cos(-60°) & -\sin(-60°) & 4 \\ \sin(-60°) & \cos(-60°) & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.500 & 0.866 & 4 \\ -0.866 & 0.500 & 4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 6.366 \\ 1.902 \\ 1 \end{pmatrix}\end{aligned}$$

Compound Transformation Matrices

The power of this formulation is that we can combine multiple rigid transformations into a single matrix. Consider the series of reference frames shown in Fig. 9. Suppose that the position and orientation of each

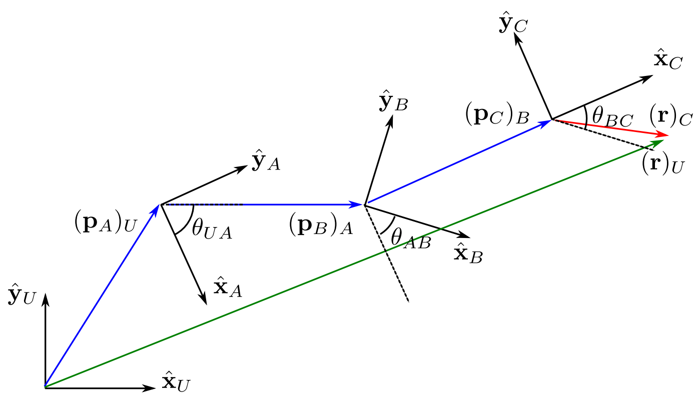


Figure 9: A series of rigid transformations

reference frame's origin is known relative to the position of the previous one and that we may compute $\mathbf{T}_A^U$, $\mathbf{T}_B^A$ and $\mathbf{T}_C^B$. Then to convert $(\mathbf{r})_C$ into $(\mathbf{r})_U$ we work our way through all of the intermediate frames, beginning with the conversion from F_C to F_B

$$(\mathbf{r})_B = \mathbf{T}_C^B(\mathbf{r})_C \tag{17}$$

then from F_B to F_A

$$(\mathbf{r})_A = \mathbf{T}_B^A(\mathbf{r})_B \tag{18}$$

and finally from F_A to F_U

$$(\mathbf{r})_U = \mathbf{T}_A^U(\mathbf{r})_A \tag{19}$$

If we substitute Eqs. 17 and 18 into Eq.19 we obtain:

$$\begin{aligned}
(\mathbf{r})_U &= \mathbf{T}_A^U(\mathbf{r})_A \\
(\mathbf{r})_U &= \mathbf{T}_A^U[\mathbf{T}_B^A(\mathbf{r})_B] \\
(\mathbf{r})_U &= \mathbf{T}_A^U[\mathbf{T}_B^A[\mathbf{T}_C^B(\mathbf{r})_C]] \\
(\mathbf{r})_U &= \underbrace{(\mathbf{T}_A^U\mathbf{T}_B^A\mathbf{T}_C^B)}_{\mathbf{T}_C^U}(\mathbf{r})_C
\end{aligned}$$

The three inner transformation matrices multiple to give a conversion from F_C to F_U thus $\mathbf{T}_A^U\mathbf{T}_B^A\mathbf{T}_C^B = \mathbf{T}_C^U$. Similarly, any number of rigid transformations can be combined. This is a very elegant and compact way of expressing a compound rigid transformation.

Example: Compound Transformation Matrices
Let's return to the example in Fig. 9. In the previous discussion we assumed that $\mathbf{T}_A^U$, $\mathbf{T}_B^A$ and $\mathbf{T}_C^B$ were given. Suppose that the relative position of each frame $(\mathbf{p}_C)_B$, $(\mathbf{p}_B)_A$ and $(\mathbf{p}_A)_U$ is known and that the angle of each frame *relative to the previous one* is also given as shown in Fig. 9. Specifically, assume the following data

$$(\mathbf{p}_C)_B = \begin{pmatrix} 2.0479 \\ 1.4339 \\ 1 \end{pmatrix} \qquad (\mathbf{p}_B)_A = \begin{pmatrix} 1.5000 \\ 2.5981 \\ 1 \end{pmatrix} \qquad (\mathbf{p}_A)_U = \begin{pmatrix} 1.5000 \\ 2.5981 \\ 1 \end{pmatrix}$$

and the relative angles are $\theta_{UA} = -60°$, $\theta_{AB} = 45°$, and $\theta_{BC} = 40°$. Note that $(\mathbf{p}_B)_A$ and $(\mathbf{p}_A)_U$ are numerically the same but they point in different directions because they are expressed in different reference frames. The position vector in the F_C frame is

$$(\mathbf{r})_C = \begin{pmatrix} 0.8 \\ -0.6 \\ 1 \end{pmatrix}$$

We wish to express this vector in the F_U frame. We will begin by working our way backwards from the F_C frame, to the F_B frame, to the F_A frame, and finally to the F_U frame. Beginning with the transformation from the F_C frame to the F_B frame, the relative angle is $\theta_{BC} = 40°$ and using $(\mathbf{p}_C)_B$

given above the transformation matrix is

$$\mathbf{T}_C^B (= \begin{pmatrix} \cos(40^\circ) & -\sin(40^\circ) & 2 \\ \sin(40^\circ) & \cos(40^\circ) & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.7660 & -0.6428 & 2.0479 \\ 0.6428 & 0.7660 & 1.4339 \\ 0 & 0 & 1 \end{pmatrix} \tag{20}$$

Similarly, from F_B to F_A we have $\theta_{AB} = 45^\circ$ and with $(\mathbf{p}_B)_A$ we obtain

$$\mathbf{T}_B^A (= \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) & 1.5000 \\ \sin(45^\circ) & \cos(45^\circ) & 2.5981 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.7071 & -0.7071 & 1.5000 \\ 0.7071 & 0.7071 & 2.5981 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

Lastly, from F_A to F_U we have $\theta_{UA} = -60^\circ$ and with $(\mathbf{p}_A)_U$ we obtain

$$\mathbf{T}_A^U (= \begin{pmatrix} \cos(-60^\circ) & -\sin(-60^\circ) & 1.5000 \\ \sin(-60^\circ) & \cos(-60^\circ) & 2.5981 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.5000 & 0.8660 & 1.5000 \\ -0.8660 & 0.5000 & 2.5981 \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

The compound transformation matrix is

$$\begin{aligned} \mathbf{T}_C^U &= \mathbf{T}_A^U \mathbf{T}_B^A \mathbf{T}_C^B \\ &= \begin{pmatrix} 0.7660 & -0.6428 & 2.0479 \\ 0.6428 & 0.7660 & 1.4339 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.7071 & -0.7071 & 1.5000 \\ 0.7071 & 0.7071 & 2.5981 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5000 & 0.8660 & 1.5000 \\ -0.8660 & 0.5000 & 2.5981 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.9063 & -0.4226 & 6.8492 \\ 0.4226 & 0.9063 & 3.4531 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and the desired vector in the F_U frame is

$$\begin{aligned} (\mathbf{r})_U &= \mathbf{T}_C^U (\mathbf{r})_C \\ &= \begin{pmatrix} 0.9063 & -0.4226 & 6.8492 \\ 0.4226 & 0.9063 & 3.4531 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 \\ -0.6 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 7.8278 \\ 3.2474 \\ 1 \end{pmatrix} \end{aligned}$$

References

[1] J. J. Craig. *Introduction to Robotics: Mechanics and Control.* Addison-Wesley, 1989.